

# ON THE RADICAL OF ENDOMORPHISM RINGS OF LOCAL MODULES

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ABSTRACT. We study the construction and properties of modules whose endomorphism rings have a unique two-sided maximal ideal.

## 1. INTRODUCTION

Let  $(\mathbf{R}, \mathfrak{m})$  be a Noetherian local ring and  $E$  a finitely generated  $\mathbf{R}$ -module. The ring  $\mathbf{C} = \text{Hom}_{\mathbf{R}}(E, E) = \text{End}(E, E)$  expresses many properties of  $E$ , and sometimes of  $\mathbf{R}$  itself, many of which are well hidden. Let us list some of these properties and related questions:

- The HomAB question: Can the number of generators of  $\mathbf{C}$  be estimated in terms of properties of  $E$ ? For a detailed discussion of this question we refer to [5], [6]. Recently, Kia Dalili resolved one its main problems in proving the existence, in the graded case, of *a priori* uniform bounds for  $\nu(\mathbf{C})$  in terms of extended cohomological degrees of  $E$ .
- Non-commutative desingularization: There are noteworthy rings of endomorphisms in the recent literature. One class of the most intriguing rigs are those of finite global dimension; see [8], [13], [14], [15].
- Degree representation of a module question: For a module  $E$ , its degree representation is the smallest integer  $r$ —when it exists—such that there is an embedding of  $\mathbf{R}$ -algebras  $\varphi : \text{Hom}_{\mathbf{R}}(E, E) \rightarrow M_r(\mathbf{R})$ . Which modules admit such a degree?
- Construction of local modules: The terminology *local module* refers to a finitely generated module  $E$  over a local Noetherian ring  $(\mathbf{R}, \mathfrak{m})$  such that the endomorphism ring  $\text{End}_{\mathbf{R}}(E) = \text{Hom}_{\mathbf{R}}(E, E)$  has a unique two-sided maximal ideal. How to build them?

We want to study some of these questions by examining the Jacobson radical of  $\mathbf{C}$ . For simplicity we often denote the module of homomorphisms  $\text{Hom}_{\mathbf{R}}(E, F)$  by  $\text{Hom}(E, F)$  and employ the similar notation for some derived functors. We will argue that local modules are key blocks in building other modules whose endomorphism rings allow for the determination of their Jacobson radicals.

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In sections 2 and 3 we develop techniques to tell when modules of syzygies are local. The most interesting classes of such modules avail themselves of the notion of the Auslander dual ([2]). Section 4 uses the notion of *small homomorphism* to study the decomposition of  $\mathbf{C}$ . The main results of this note are given in the last section, particularly in Theorem 5.1, showing that if  $\mathbf{R}$  is normal, contains the rationals and  $E$  torsionfree, then if  $E$  is a local module that is  $\mathbf{C}$ -projective then  $E$  is  $\mathbf{R}$ -free.

## 2. JACOBSON RADICAL

In this section and next, we treat conditions for the algebra  $\mathbf{C} = \text{Hom}_{\mathbf{R}}(E, E)$  to have a unique two-sided maximal ideal. An example is a free  $\mathbf{R}$ -module. Observe that by Nakayama Lemma,  $\mathbf{m}\mathbf{C}$  is contained in the Jacobson radical  $\mathbf{J}$ . We will identify other subideals of  $\mathbf{J}$ .

We will denote the  $\mathbf{R}$ -dual of  $E$  by  $E^* = \text{Hom}_{\mathbf{R}}(E, \mathbf{R})$ , and say that  $E$  is reflexive if the usual mapping  $E \rightarrow E^{**}$  is an isomorphism. These modules admit a natural  $\mathbf{C}$  action,  $E$  as a left  $\mathbf{C}$ -module and  $E^*$  as a right  $\mathbf{C}$ -module.

**Proposition 2.1.** *Let  $(\mathbf{R}, \mathbf{m})$  be a Noetherian local ring and let  $E$  be a finitely generated  $\mathbf{R}$ -module.*

- (i) *If  $E$  has no free summand, then the image of  $E^* \otimes E$  in  $\mathbf{C}$  is a two-sided ideal contained in the Jacobson radical.*
- (ii)  *$\text{Hom}_{\mathbf{R}}(E, \mathbf{m}E)$  is a two-sided ideal contained in  $\mathbf{J}$ .*

*Proof.* There are two actions of  $\mathbf{C}$  on  $E^* \otimes E$ . For  $\mathbf{h} \in \mathbf{C}$ ,  $(f \otimes e)\mathbf{h} = f \circ \mathbf{h} \otimes e$  and  $\mathbf{h}(f \otimes e) = f \otimes \mathbf{h}(e)$ .

Let  $\mathbf{I}$  be the identity of  $\mathbf{C}$ . To prove that

$$\mathbf{h} = \mathbf{I} + \sum_{i=1}^n f_i \otimes e_i$$

is invertible, note that for each  $e \in E$ ,

$$\mathbf{h}(e) = e + \sum_{i=1}^n f_i(e)e_i \in e + \mathbf{m}E,$$

since  $f_i(e) \in \mathbf{m}$  as  $E$  has no free summand. From the Nakayama Lemma, it follows that  $\mathbf{h}$  is a surjective endomorphism, and therefore must be invertible.

The proof of (ii) is similar. □

Throughout  $(\mathbf{R}, \mathbf{m})$  is a Noetherian local ring and  $E$  is a finitely generated  $\mathbf{R}$ -module. The ideal  $E^*(E) = \tau(E) = (f(e), f \in E^*, e \in E)$  is the *trace ideal* of  $E$ . It is of interest to describe classes of modules of syzygies with  $\tau(E) \subset \mathbf{m}$ . If the trace ideal  $E^*(E) = \mathbf{R}$ , by Morita's theory (see [3, Theorem A.2]),  $E$  is a projective module over

$\mathbf{C} = \text{End}(E)$ . This is easy to verify directly as follows. Suppose  $E = \mathbf{R}\epsilon \oplus M$ . Then  $\mathbf{C} = \text{End}(E)$  may be represented as

$$\mathbf{C} = \begin{bmatrix} R & M^* \\ M & \text{End}(M) \end{bmatrix}.$$

As a left  $\mathbf{C}$ -module,  $E = \mathbf{C}\epsilon$  where the annihilator of  $\epsilon$  is the left ideal

$$L = \begin{bmatrix} 0 & M^* \\ 0 & \text{End}(M) \end{bmatrix},$$

which splits off  $\mathbf{C}$ .

**Remark 2.2.** There are cases however when  $E$  is  $\mathbf{C}$ -projective but  $E^*(E) \neq \mathbf{R}$ . For instance, if  $\mathbf{R}$  is a  $k[[x, y]]/(y^2 + x^2 + y^3)$ , then any ideal has this property.

**Remark 2.1.** From the sequence

$$0 \rightarrow \mathfrak{m}E \rightarrow E \rightarrow E/\mathfrak{m}E = \overline{E} = k^n \rightarrow 0,$$

we have the exact sequence

$$0 \rightarrow \text{Hom}_{\mathbf{R}}(E, \mathfrak{m}E) \rightarrow \mathbf{C} \rightarrow M_n(k) \rightarrow \text{Ext}_{\mathbf{R}}(E, \mathfrak{m}E).$$

This shows that  $\dim \mathbf{C}/\mathbf{J} \leq n^2$ , since  $\text{Hom}_{\mathbf{R}}(E, \mathfrak{m}E) \subset \mathbf{J}$ , giving an explicit control over the number of maximal two-sided ideals of  $\mathbf{C}$ .

If  $E$  is a module without free summand, we use the notation of the subideals of  $\mathbf{J}$

$$\mathbf{J}_1 = \text{image } E^* \otimes E \subset \mathbf{J}_0 = \text{Hom}(E, \mathfrak{m}E) \subset \mathbf{J}.$$

We introduce some terminology on special homomorphisms.

**Definition 2.3.** Let  $(\mathbf{R}, \mathfrak{m})$  be Noetherian local ring, and let  $E$  and  $F$  be finitely generated  $\mathbf{R}$ -modules. A homomorphism  $\varphi : E \rightarrow F$  is said to be *small* if  $\varphi(E) \subset \mathfrak{m}F$ .

In assembling the Jacobian ideal of the ring of endomorphisms of direct sums we will consider conditions such as:

- All homomorphisms in  $\text{Hom}(E, F)$  are small.
- The endomorphisms of  $E$  that factor through  $F$  are small, that is

$$\text{Hom}(F, E) \circ \text{Hom}(E, F) \subset \text{Hom}(E, \mathfrak{m}E).$$

**Modules without free summands.** Let  $(\mathbf{R}, \mathfrak{m})$  be a Cohen–Macaulay local ring of dimension  $d$  and  $E$  a finitely generated  $\mathbf{R}$ -module.

**Example 2.4.** We will now describe two classes of modules of syzygies that do not have free summands.

- (1) [10, Lemma 1.4] Let  $\mathbf{R}$  be a Cohen–Macaulay local ring and  $E$  a module of syzygies

$$0 \rightarrow E \rightarrow F \rightarrow M \rightarrow 0,$$

$E \subset \mathfrak{m}F$  and  $M$  a maximal Cohen–Macaulay module. Then  $E$  has no free summand.

*Proof.* Suppose  $E = \mathbf{R}\epsilon \oplus E'$ ,  $\epsilon = (a_1, \dots, a_n) \in \mathfrak{m}F$ . Setting  $M' = F/\mathbf{R}\epsilon$  we have the exact sequence

$$0 \rightarrow E' \rightarrow M' \rightarrow M \rightarrow 0$$

showing that  $M'$  is a maximal Cohen–Macaulay module.

Let  $\mathbf{x}$  be a system of parameters. Tensoring  $M'$  by  $\overline{\mathbf{R}} = \mathbf{R}/(\mathbf{x})$ , we get the exact sequence

$$0 \rightarrow \overline{\mathbf{R}}\epsilon \rightarrow F \otimes \overline{\mathbf{R}} \rightarrow M' \otimes \overline{\mathbf{R}} \rightarrow 0.$$

But this is impossible since the entries of  $\epsilon$  are zero divisors of the Artin ring  $\overline{\mathbf{R}}$  (or directly by the Auslander–Buchsbaum equality). □

- (2) Suppose  $\mathbf{R}$  is a Gorenstein local ring and  $E$  is a module of syzygies

$$0 \rightarrow E \xrightarrow{\varphi} F_s \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0,$$

$E \subset \mathfrak{m}F_s$ . If  $1 \leq s \leq \text{height}(\text{ann}(M)) - 2$  then  $E$  has no free summand.

*Proof.*  $E$  is a reflexive being a second syzygy module over the Gorenstein ring  $\mathbf{R}$ . It will be enough to show the dual module of  $\mathbf{R}\epsilon$  splits off  $F_s^*$ .

By assumption  $\text{Ext}_{\mathbf{R}}^i(M, \mathbf{R}) = 0$  for  $i \leq s + 1$ . Applying  $\text{Hom}(\cdot, \mathbf{R})$  to the complex we obtain an exact complex

$$0 \rightarrow F_0^* \rightarrow F_1^* \rightarrow \dots \rightarrow F_s^* \xrightarrow{\varphi^*} E^* \rightarrow 0,$$

to prove the assertion since  $\varphi = \varphi^{**}$ . □

The assumption that  $\mathbf{R}$  is Gorenstein was used to guarantee that second syzygies modules are reflexive. This could be achieved in many other ways. For example, with  $\mathbf{R}$  Cohen–Macaulay and  $\text{height ann}(M) \geq 2$ .

- (3) Taken together these two observations give an overall picture of the syzygies of  $M$  which are without free summands.
- (4) A special case is that of the syzygies in the Koszul complex of a regular sequence.

## 3. CONSTRUCTION OF LOCAL MODULES

**Construction of endomorphisms.** For an  $\mathbf{R}$ -module  $A$ , its ring of endomorphisms  $\text{End}_{\mathbf{R}}(A)$  arise out of the syzygies of  $A$  as follows.

**Proposition 3.1.** *Let  $A$  be a finitely generated  $\mathbf{R}$ -module with a presentation*

$$\mathbf{R}^m \xrightarrow{\varphi} \mathbf{R}^n \longrightarrow A \rightarrow 0.$$

*Choose bases  $\{e_1, \dots, e_n\}$  and  $\{f_1, \dots, f_m\}$  in  $\mathbf{R}^n$  and  $\mathbf{R}^m$ . Then  $\text{Hom}_{\mathbf{R}}(A, A)$  is isomorphic to the kernel of the induced mapping*

$$\Phi_A : A \otimes_{\mathbf{R}} (\mathbf{R}^n)^* \xrightarrow{\text{I} \otimes \varphi^*} A \otimes (\mathbf{R}^m)^*.$$

*Proof.* We let  $e_i^*$  be the corresponding dual basis of  $(\mathbf{R}^n)^*$ . An element  $z = \sum x_i \otimes e_i^*$  lies in the kernel of  $\Phi$  if  $\sum x_i \otimes e_i^* \circ \varphi^* = 0$ , a condition that means  $[x_1, \dots, x_n] \cdot \varphi^* = 0$ . This will imply that if we let  $a_i$  be the image of  $e_i$  in  $A$ , the assignment  $a_i \rightarrow x_i$  will define an endomorphism  $\alpha : A \rightarrow A$ .

Conversely, given such  $\alpha$ ,  $\zeta = \sum_i \alpha(a_i) \otimes e_i^* \in \ker(\Phi_A)$ . The mappings are clearly inverses of one another.  $\square$

**Auslander dual.** Our main vehicle to build local modules out of other local modules is the following notion of Maurice Auslander ([2]).

**Definition 3.2.** Let  $E$  be a finitely generated  $\mathbf{R}$ -module with a projective presentation

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow E \rightarrow 0.$$

The *Auslander dual* of  $E$  is the module  $D(E) = \text{coker}(\varphi^*)$ ,

$$(3.0.1) \quad 0 \rightarrow E^* \longrightarrow F_0^* \xrightarrow{\varphi^*} F_1^* \longrightarrow D(E) \rightarrow 0.$$

The module  $D(E)$  depends on the chosen presentation but it is unique up to projective summands. In particular the values of the functors  $\text{Ext}_{\mathbf{R}}^i(D(E), \cdot)$  and  $\text{Tor}_{\mathbf{R}}^i(D(E), \cdot)$ , for  $i \geq 1$ , are independent of the presentation. Its use here lies in the following result (see [2, Chapter 2]):

**Proposition 3.3.** *Let  $\mathbf{R}$  be a Noetherian ring and  $E$  a finitely generated  $\mathbf{R}$ -module. There are two exact sequences of functors:*

$$(3.0.2) \quad 0 \rightarrow \text{Ext}_{\mathbf{R}}^1(D(E), \cdot) \longrightarrow E \otimes_{\mathbf{R}} \cdot \longrightarrow \text{Hom}_{\mathbf{R}}(E^*, \cdot) \longrightarrow \text{Ext}_{\mathbf{R}}^2(D(E), \cdot) \rightarrow 0$$

$$(3.0.3) \quad 0 \rightarrow \text{Tor}_{\mathbf{R}}^2(D(E), \cdot) \longrightarrow E^* \otimes_{\mathbf{R}} \cdot \longrightarrow \text{Hom}_{\mathbf{R}}(E, \cdot) \longrightarrow \text{Tor}_{\mathbf{R}}^1(D(E), \cdot) \rightarrow 0.$$

The setup we employ is derived from the analysis of duality of [2]: There is an exact complex of  $\mathbf{R}$ -algebras

$$(3.0.4) \quad E^* \otimes_{\mathbf{R}} E \longrightarrow \text{Hom}_{\mathbf{R}}(E, E) = \text{End}_{\mathbf{R}}(E) \longrightarrow \text{Tor}_{\mathbf{R}}^1(D(E), E) \rightarrow 0$$

where  $D(E)$  is the Auslander dual of  $E$ . The question turns on the understanding of  $\text{Tor}_{\mathbf{R}}^1(D(E), E)$ , a module that by [11, Theorem 1.3] is identified with the

module of natural endomorphisms  $\underline{\text{Hom}}_{\mathbf{R}}(\text{Ext}_{\mathbf{R}}^1(E, \cdot), \text{Ext}_{\mathbf{R}}^1(E, \cdot))$ . In several cases,  $\text{Tor}_1^{\mathbf{R}}(D(E), E)$  is actually identified to another ring of endomorphisms  $\text{Hom}_{\mathbf{R}}(M, M)$ , with  $M$  having much smaller support than  $E$  and but still explicitly related to  $E$ .

For each application of this setup we look for which of  $\underline{\text{Hom}}_{\mathbf{R}}(\text{Ext}_{\mathbf{R}}^1(E, \cdot), \text{Ext}_{\mathbf{R}}^1(E, \cdot))$ .  $\text{Tor}_1^{\mathbf{R}}(D(E), E)$  is more amenable. Let us consider two examples.

**Example 3.1.** Let  $\mathbf{R}$  be a Cohen–Macaulay local ring of dimension  $d \geq 3$ , and let  $E$  be a module defined by one relation

$$0 \rightarrow \mathbf{R} \xrightarrow{\varphi} \mathbf{R}^n \rightarrow E \rightarrow 0.$$

We assume that the entries of  $\varphi$  define an ideal  $I$  of height  $\geq 3$ , minimally generated by  $n$  elements. Let us determine the number of generators of  $\mathbf{C} = \text{Hom}_{\mathbf{R}}(E, E)$ , and some of its other properties.

The functor  $\text{Ext}_{\mathbf{R}}^1(E, X) = \text{Ext}_{\mathbf{R}}^1(E, \mathbf{R}) \otimes X = \mathbf{R}/I \otimes X$ , so that

$$\underline{\text{Hom}}_{\mathbf{R}}(\text{Ext}_{\mathbf{R}}^1(E, \cdot), \text{Ext}_{\mathbf{R}}^1(E, \cdot)) = \text{Hom}_{\mathbf{R}}(\mathbf{R}/I, \mathbf{R}/I) = \mathbf{R}/I.$$

It is also clear that  $E^* \otimes E$  is torsionfree and that  $E^*$  has no free summand. By Proposition 2.1 it follows that  $E$  is a local module. As for the number of generators of  $\mathbf{C}$ ,

$$\beta_0(I)\beta_1(I) - \beta_0(I) + 1 \leq \nu(\mathbf{C}) \leq \nu(E)\nu(E^*) + 1 = \beta_0(I)\beta_1(I) + 1.$$

**Example 3.2.** Let  $E$  be a module with a free resolution

$$0 \rightarrow F_n \xrightarrow{\psi} F_{n-1} \rightarrow \cdots \rightarrow F_1 \xrightarrow{\varphi} F_0 \rightarrow E \rightarrow 0.$$

Assume that  $\text{Ext}_{\mathbf{R}}^i(E, \mathbf{R}) = 0$ ,  $1 \leq i \leq n-1$ . Dualizing we have the exact complex

$$0 \rightarrow E^* \rightarrow F_0^* \xrightarrow{\varphi^*} F_1^* \rightarrow \cdots \rightarrow F_{n-1}^* \xrightarrow{\psi^*} F_n^* \rightarrow \text{Ext}_{\mathbf{R}}^n(E, \mathbf{R}) \rightarrow 0.$$

This gives that  $D(E) = \text{coker } \varphi^*$  and thus  $\text{Tor}_1^{\mathbf{R}}(D(E), E) = \text{Tor}_n^{\mathbf{R}}(\text{Ext}_{\mathbf{R}}^n(E, \mathbf{R}), E)$ . Note also that  $D(\text{Ext}_{\mathbf{R}}^n(E, \mathbf{R})) = \text{coker } \psi$ . Now we apply Proposition 3.1 to the module  $A = \text{Ext}_{\mathbf{R}}^n(E, \mathbf{R})$  to get isomorphisms

$$\text{Hom}(\text{Ext}_{\mathbf{R}}^n(E, \mathbf{R}), \text{Ext}_{\mathbf{R}}^n(E, \mathbf{R})) = \ker \Phi_A = \text{Tor}_n^{\mathbf{R}}(\text{Ext}_{\mathbf{R}}^n(E, \mathbf{R}), E).$$

This gives the exact sequence

$$0 \rightarrow E^* \otimes E \rightarrow \text{End}(E) \rightarrow \text{End}(\text{Ext}_{\mathbf{R}}^n(E, \mathbf{R})) \rightarrow 0,$$

since  $\text{Tor}_2^{\mathbf{R}}(D(E), E) = \text{Tor}_{n+1}^{\mathbf{R}}(\text{Ext}_{\mathbf{R}}^n(E, \mathbf{R}), E) = 0$ .

**Syzygies of perfect modules.** Let  $\mathbf{R}$  be a Gorenstein local ring of dimension  $d$ . Let us consider some modules with a very rich structure—the modules of syzygies of Cohen-Macaulay modules, or of mild generalizations thereof.

Let us begin this discussion with an example, the modules of cycles of a Koszul complex  $\mathbf{K}(\mathbf{x})$  associated to a regular sequence  $\mathbf{x} = \{x_1, \dots, x_n\}$ ,  $n \geq 5$ :

$$\mathbf{K}(\mathbf{x}) : 0 \rightarrow K_n \rightarrow K_{n-1} \rightarrow K_{n-2} \rightarrow \cdots \rightarrow K_2 \rightarrow K_1 \rightarrow K_0 \rightarrow 0.$$

For simplicity we take for module  $E$  the 1-cycles  $Z_1$  of  $\mathbf{K}$ . There is a pairing in the subalgebra  $\mathbf{Z}$  of cycles leading to

$$Z_1 \times Z_{n-2} \rightarrow Z_{n-1} = \mathbf{R},$$

that identifies  $Z_{n-2}$  with the dual  $E^*$  of  $E$ .

We are now ready to put this data into the framework of the Auslander dual. Dualizing the projective presentation of  $E$ ,

$$0 \rightarrow K_n \longrightarrow \cdots \longrightarrow K_4 \longrightarrow K_3 \longrightarrow E = Z_2 \rightarrow 0,$$

gives us the exact complex

$$0 \rightarrow E^* \longrightarrow K_3^* \longrightarrow K_4^* \longrightarrow D(E) \rightarrow 0.$$

In other words, to the identification  $D(E) = Z_{n-3}$ :

$$0 \rightarrow Z_{n-3} \longrightarrow K_{n-3} \longrightarrow K_{n-4} \longrightarrow Z_{n-5} \rightarrow 0.$$

Now for the computation of  $\text{Tor}_1^{\mathbf{R}}(D(E), E)$ :

$$\begin{aligned} \text{Tor}_1^{\mathbf{R}}(D(E), E) &= \text{Tor}_1^{\mathbf{R}}(Z_{n-5}, E) = \text{Tor}_2^{\mathbf{R}}(Z_{n-6}, E) = \cdots \\ &= \text{Tor}_{n-4}^{\mathbf{R}}(Z_0, E) = \text{Tor}_{n-3}^{\mathbf{R}}(\mathbf{R}/I, E) = K_n \otimes \mathbf{R}/I = \mathbf{R}/I. \end{aligned}$$

**Remark 3.4.** The number of generators of  $\text{Hom}_{\mathbf{R}}(E, E)$  is bounded by

$$\nu(E)\nu(E^*) + 1.$$

For the purpose of a comparison, let us evaluate  $\text{hdeg}(E)$  (for which refer to [5]). The multiplicity of  $E$  is  $(n-1)\deg(\mathbf{R})$ . Applying  $\text{Hom}_{\mathbf{R}}(\cdot, R)$  to the projective resolution of  $E$ , we get  $\text{Ext}_{\mathbf{R}}^{n-2}(E, \mathbf{R}) = \text{Ext}_{\mathbf{R}}^n(\mathbf{R}/I, \mathbf{R}) = \mathbf{R}/I$  is Cohen-Macaulay, and its contribution in the formula for  $\text{hdeg}(E)$  becomes

$$\text{hdeg}(E) = \deg(E) + \binom{d-1}{n-3} \text{hdeg}(\text{Ext}_{\mathbf{R}}^{n-2}(E, \mathbf{R})) = (n-1) + \binom{d-1}{n-3} \deg(\mathbf{R}/I).$$

It is clear that in appealing to [5, Theorem 5.2], to get information about  $\nu(E^*)$ , a similar calculation can be carried out for any module of cycles of a projective resolution of broad classes of Cohen-Macaulay modules.

Let  $(\mathbf{R}, \mathfrak{m})$  be a Gorenstein local ring of dimension  $d$ . Let  $M$  be a perfect  $\mathbf{R}$ -module with a minimal free resolution

$$\mathbf{K} : 0 \rightarrow K_n \rightarrow K_{n-1} \rightarrow K_{n-2} \rightarrow \cdots \rightarrow K_2 \rightarrow K_1 \rightarrow K_0 \rightarrow M \rightarrow 0.$$

We observe that dualizing  $\mathbf{K}$  gives a minimal projective resolution  $\mathbf{L}$  of  $\text{Ext}_{\mathbf{R}}^n(M, \mathbf{R})$ . Let  $E$  be the module  $Z_{k-1} = Z_{k-1}(\mathbf{K})$  of  $k-1$ -cycles of  $\mathbf{K}$ ,

$$K_{k+1} \longrightarrow K_k \longrightarrow E \rightarrow 0.$$

Dualizing, to define the Auslander dual  $D(E)$ , gives the complex

$$0 \rightarrow E^* \longrightarrow L_{n-k} \longrightarrow L_{n-k-1} \longrightarrow D(E) \rightarrow 0.$$

It identifies  $E^*$  with the  $(n-k+1)$ -cycles of  $\mathbf{L}$ , and  $D(E)$  with its  $(n-k-2)$ -cycles. In particular this gives  $\nu(E) = \beta_k(M)$  and  $\nu(E^*) = \beta_{k-1}(M)$ .

Now to make use of the Auslander dual setup, we seek some control over  $\text{Tor}_1^{\mathbf{R}}(D(E), E)$ . We make use first of the complex

$$0 \rightarrow D(E) \longrightarrow L_{n-k-2} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow \text{Ext}_{\mathbf{R}}^n(M, \mathbf{R}) \rightarrow 0,$$

to get

$$\text{Tor}_1^{\mathbf{R}}(D(E), E) \simeq \text{Tor}_{n-k}^{\mathbf{R}}(\text{Ext}_{\mathbf{R}}^n(M, \mathbf{R}), E),$$

and then of the minimal resolution of  $E$  to obtain

$$\text{Tor}_1^{\mathbf{R}}(D(E), E) \simeq \text{Tor}_n^{\mathbf{R}}(\text{Ext}_{\mathbf{R}}^n(M, \mathbf{R}), M).$$

In particular,  $\text{Tor}_1^{\mathbf{R}}(D(E), E)$  is independent of which module of syzygies was taken. Furthermore, the calculation shows that  $\text{Tor}_2^{\mathbf{R}}(D(E), E) = 0$ .

Placing these elements together, we have the exact sequence

$$0 \rightarrow E^* \otimes E \longrightarrow \text{Hom}_{\mathbf{R}}(E, E) \longrightarrow \text{Tor}_n^{\mathbf{R}}(\text{Ext}_{\mathbf{R}}^n(M, \mathbf{R}), M) \rightarrow 0.$$

Note that  $E^* \otimes E$  is a torsion free  $\mathbf{R}$ -module.

Finally, it follows from Proposition 3.1 that  $\text{Tor}_n^{\mathbf{R}}(\text{Ext}_{\mathbf{R}}^n(M, \mathbf{R}), M)$  can be identified to  $\text{Hom}_{\mathbf{R}}(M, M)$ .

Let us sum up these observations in the following:

**Theorem 3.5.** *Let  $\mathbf{R}$  be a Gorenstein local ring and let  $M$  be a perfect module with a minimal resolution  $\mathbf{K}$ . For the module  $E$  of  $k$ -syzygies of  $M$ , there exists an exact sequence*

$$0 \rightarrow E^* \otimes E \longrightarrow \text{Hom}_{\mathbf{R}}(E, E) \longrightarrow \text{Hom}_{\mathbf{R}}(M, M) \rightarrow 0.$$

This gives the bound

$$\nu(\text{Hom}_{\mathbf{R}}(E, E)) \leq \beta_k(M)\beta_{k-1}(M) + \nu(\text{Hom}_{\mathbf{R}}(M, M)).$$

If  $M$  is cyclic, or  $\dim M = 2$ , the estimation is easy. The formula also shows up the case of MCM modules to be a corner case for the general HomAB problem.



**Theorem 3.6.** *Let  $\mathbf{R}$  be a Gorenstein local ring and let  $M$  be a perfect module with a minimal resolution. For the module  $E$  of  $k$ -syzygies of  $M$ ,  $k < \text{proj dim } M$ , there exists an exact sequence*

$$0 \rightarrow E^* \otimes E \rightarrow \text{Hom}_{\mathbf{R}}(E, E) \rightarrow \text{Hom}_{\mathbf{R}}(M, M) \rightarrow 0.$$

**Corollary 3.7.** *If  $M = \mathbf{R}/I$ ,  $E$  is a local module.*

**Corollary 3.8.** *Suppose  $\mathbf{R}$  is a Gorenstein local ring of dimension  $d$ . If  $M = \mathbf{R}/I$  is a Cohen-Macaulay module of projective dimension  $n$  and  $E$  is a module of  $k$ -syzygies of  $M$  for  $k < n$ , then*

$$\text{depth End}(E) \geq d - n + 1.$$

*Proof.* It will be enough to show that  $\text{depth } E \otimes E^* \geq d - n + 1$ .

Dualizing the presentation

$$0 \rightarrow E \rightarrow K_{k-1} \rightarrow \cdots \rightarrow K_1 \rightarrow K_0 \rightarrow M \rightarrow 0,$$

we obtain a projective resolution of  $E^*$

$$0 \rightarrow K_0^* \rightarrow K_1^* \rightarrow \cdots \rightarrow K_{k-1}^* \rightarrow E^* \rightarrow 0.$$

Note that  $\text{depth } E = d - n + k$  and  $\text{depth } E^* = d - k + 1$ . Tensoring the second complex by  $E$  we get a complex of modules of depth  $d - n + k$ . It follows easily that the complex is exact, that thus  $\text{depth } E \otimes E^* \geq (d - n + k) - (k - 1) = d - n + 1$ .  $\square$

**Some generic modules.** Let  $\varphi$  be a generic  $n \times m$  matrix over a field  $k$ ,  $\varphi = [\mathbf{x}] = [x_{ij}]$ ,  $1 \leq i \leq m, 1 \leq j \leq n$ ,  $m \geq n$ , set  $\mathbf{R} = k[\mathbf{x}]_{(\mathbf{x})}$  and consider the module  $E$

$$0 \rightarrow \mathbf{R}^n \xrightarrow{\varphi} \mathbf{R}^m \rightarrow E \rightarrow 0.$$

We study the ring  $\mathbf{C} = \text{Hom}_{\mathbf{R}}(E, E)$ . Let  $I = I_n(\varphi)$  be the ideal of maximal minors of  $\varphi$ .  $\mathbf{R}/I$  is a normal Cohen-Macaulay domain and many of its properties are deduced from the Eagon-Northcott complexes ([9]). We are going to use them to obtain corresponding properties of  $E$ :

- If  $n < m \geq 3$  (which we assume throughout),  $E$  is a reflexive module.
- By dualizing we obtain the Auslander dual  $D(E)$  of  $E$ :

$$0 \rightarrow E^* \rightarrow \mathbf{R}^m \xrightarrow{\varphi^*} \mathbf{R}^n \rightarrow D(E) = \text{Ext}_{\mathbf{R}}^1(E, \mathbf{R}) \rightarrow 0.$$

- $\text{Ext}_{\mathbf{R}}^1(E, \mathbf{R})$  is isomorphic to an ideal of  $\mathbf{R}/I$ :  $\text{Ext}_{\mathbf{R}}^1(E, \mathbf{R})$  is a perfect  $\mathbf{R}$ -module, annihilated  $I$ . These assertions follow directly from the perfection of the complexes. Finally, localizing at  $I$  the complex is quasi-isomorphic to Koszul complex of a regular sequence which yield

$$\text{Ext}_{\mathbf{R}}^1(E, \mathbf{R})_I \simeq (\mathbf{R}/I)_I.$$

**Theorem 3.3.**  *$\mathbf{C} = \text{Hom}_{\mathbf{R}}(E, E)$  is a local ring generated by  $\nu(E^*)\nu(E) + 1 = n \times \binom{m}{n+1} + 1$  elements.*

*Proof.* As in the discussion of example above, The functor  $\text{Ext}_{\mathbf{R}}^1(E, X) = \text{Ext}_{\mathbf{R}}^1(E, \mathbf{R}) \otimes X = \mathbf{R}/I \otimes X$ , so that

$$\underline{\text{Hom}}_{\mathbf{R}}(\text{Ext}_{\mathbf{R}}^1(E, \cdot), \text{Ext}_{\mathbf{R}}^1(E, \cdot)) = \text{Hom}_{\mathbf{R}/I}(\text{Ext}_{\mathbf{R}}^1(E, \mathbf{R}), \text{Ext}_{\mathbf{R}}^1(E, \mathbf{R}) = \mathbf{R}/I,$$

as  $\text{Ext}_{\mathbf{R}}^1(E, \mathbf{R})$  is isomorphic to an ideal of the normal domain  $\mathbf{R}/I$ . Furthermore, since  $E$  is reflexive and  $E^*$  is a module of syzygies in the Eagon–Northcott complex it has no free summand according to a previous observation,  $E$  has the same property. The final assertion comes from the complex again.  $\square$

#### 4. DECOMPOSITION

Let  $(\mathbf{R}, \mathfrak{m})$  be a local ring and  $E$  a finitely generated  $\mathbf{R}$ -module. Suppose  $E = E_1 \oplus E_2$  is a nontrivial decomposition. We seek to express to relate the Jacobson of  $\mathbf{C} = \text{Hom}(E, E)$  to those of the subrings  $\mathbf{C}_i = \text{Hom}(E_i, E_i)$ ,  $i = 1, 2$  and the relationships between  $E_1$  and  $E_2$ . A special case is that where  $E_1 = F = \mathbf{R}^n$  and  $E_2 = M$  is a module without free summands.

**Proposition 4.1.** *Let  $\mathbf{R}$  be a local ring and  $E$  a module with a decomposition  $E = E_1 \oplus E_2$ . Set  $\mathbf{C}_i = \text{Hom}_{\mathbf{R}}(E_i, E_i)$  and  $\mathbf{J}'_i = \text{Hom}_{\mathbf{R}}(E_i, \mathfrak{m}E_i)$ . Assume the transition conditions*

$$\text{Hom}_{\mathbf{R}}(E_j, E_i) \cdot \text{Hom}_{\mathbf{R}}(E_i, E_j) \subset \text{Hom}_{\mathbf{R}}(E_j, \mathfrak{m}E_j), \quad i \neq j.$$

*Then the Jacobson radical of  $\mathbf{C} = \text{End}(E)$  is*

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \text{Hom}(E_2, E_1) \\ \text{Hom}(E_1, E_2) & \mathbf{J}_2 \end{bmatrix},$$

*where  $\mathbf{J}_i$  is the Jacobson radical of  $\text{End}(E_i)$ .*

*Proof.* Note that  $\mathbf{J}'_i$  is a subideal of  $\mathbf{J}_i$ , and for  $A = \text{Hom}(E_1, E_2)$  and  $B = \text{Hom}(E_2, E_1)$  it holds that  $A \cdot B \subset \mathbf{J}'_2$  and  $B \cdot A \subset \mathbf{J}'_1$ . From these it follows that

$$\mathbf{L} = \begin{bmatrix} \mathbf{J}_1 & B \\ A & \mathbf{J}_2 \end{bmatrix}$$

is a two-sided ideal of  $\mathbf{C}$ .

To show  $\mathbf{L}$  is the radical it is enough to show that for  $\Phi \in \mathbf{L}$ ,  $\mathbf{I} + \Phi$  is invertible, or equivalently it is surjective endomorphism of  $E$ . In other words, for  $a \in \mathbf{J}_1$ ,  $b \in A$ ,  $c \in B$  and  $d \in \mathbf{J}_2$  the system of equations

$$\begin{aligned} x + ax + by &= u \\ cx + y + dy &= v \end{aligned}$$

for  $u \in E_1$ ,  $v \in E_2$  has always a solution  $x \in E_1$  and  $y \in E_2$ . It is enough to observe that in the formal solution

$$\begin{aligned} x &= (\mathbf{I}_1 + a)^{-1}(u - by) \\ y &= (\mathbf{I}_2 + d - c(\mathbf{I}_1 + a)^{-1}b)^{-1}(v - c(\mathbf{I}_1 + a)^{-1}u), \end{aligned}$$

$$c(\mathbf{I}_1 + a)^{-1}b \in \mathbf{J}_2.$$

□

**Corollary 4.2.** *Let  $\mathbf{R}$  be a local ring and  $M$  a module without free summands. If  $F$  is a free  $\mathbf{R}$ -module of rank  $n$  then the Jacobson radical of  $\mathbf{C} = \text{End}(F \oplus M)$  is*

$$\mathbf{J} = \begin{bmatrix} \mathfrak{m} \cdot \text{End}(F) & \text{Hom}(M, F) \\ \text{Hom}(F, M) & \mathbf{J}_0 \end{bmatrix},$$

where  $\mathbf{J}_0$  is the Jacobson radical of  $\text{End}(M)$ .

This shows that a non-free module with a free summand is never local. It permits, to understand the Jacobson radical of  $\text{End}(E)$ , to peel away from  $E$  a free summand of maximal rank.

## 5. HOMOLOGICAL PROPERTIES OF LOCAL MODULES

**Projective generation.** Let  $(\mathbf{R}, \mathfrak{m})$  be a Noetherian local ring and  $E$  a finitely generated  $\mathbf{R}$ -module. Set  $\mathbf{C} = \text{End}(E)$  and consider the natural action of  $\mathbf{C}$  on  $E$ .

Let us begin with an observation.

**Proposition 5.1.** *Let  $E$  be a local module. If  $M$  is a projective  $\mathbf{C}$ -module, then  $M$  is a generator of  $\text{Mod}(\mathbf{C})$ .*

*Proof.* Let  $\mathbf{J}$  be the Jacobson radical of  $\mathbf{C}$ . Up to isomorphism there is only one indecomposable  $\mathbf{C}/\mathbf{J}$ -module, which we denote by  $L$ .

Let  $F$  be a finitely generated left  $\mathbf{C}$ -module. Then  $M/\mathbf{J}M$  and  $F/\mathbf{J}F$  are isomorphic to direct sums  $L^n$  and  $L^m$  of  $L$ . It follows that for some integer  $r > 0$  there is a surjection

$$\varphi : M^r \rightarrow F/\mathbf{J}F.$$

Since  $M$  is projective,  $\varphi$  can be lifted to a mapping  $\Phi : M^r \rightarrow F$  such that  $F = \Phi(M^r) + \mathbf{J}F$ . By Nakayama Lemma  $\Phi$  is surjective, as desired.

□

**Theorem 5.1.** *Let  $(\mathbf{R}, \mathfrak{m})$  a Noetherian local ring and  $E$  a local module. Set  $\mathbf{C} = \text{End}(E)$ .*

- (1) *If  $E$  is a projective  $\mathbf{C}$ -module then  $E$  is cyclic.*
- (2) *Moreover, suppose that  $\mathbf{R}$  is integrally closed and  $E$  is torsionfree. If the rank  $r$  of  $E$  is invertible in  $\mathbf{R}$  then  $E$  is  $\mathbf{R}$ -free.*

*Proof.* (i) Write  $E = F \oplus M$ , where  $M$  is a module of trace ideal  $\neq \mathbf{R}$ . If both  $F$  and  $M$  are nonzero then  $E$  is not a local module, according to Corollary 4.2. Assume then that  $E$  has no free summand.

By Proposition 5.1,  $E$  is a projective generator and in particular we have an isomorphism of left  $\mathbf{C}$ -modules

$$(5.1.1) \quad E^s \simeq \mathbf{C} \oplus H.$$

Since  $\mathbf{R}$  is a local ring this gives rise to a decomposition of  $\mathbf{C}$ -modules,

$$E^r \simeq \mathbf{C},$$

where  $r$  is necessarily the rank of  $E$  as an  $\mathbf{R}$ -modules. If  $\mathbf{J}$  is the Jacobson radical of  $\mathbf{C}$ , we have

$$E^r / \mathbf{J}E^r \simeq \mathbf{C} / \mathbf{J}.$$

Thus  $E / \mathbf{J}E$  is isomorphic to an ideal of the simple ring  $\mathbf{C} / \mathbf{J}$ . By Nakayama Lemma it follows that  $E$  is a cyclic  $\mathbf{C}$ -module.

To prove (ii), we recall how the *trace* of the elements of  $\mathbf{C}$  may be defined. Let  $\mathbf{S}$  be the field of fractions of  $\mathbf{R}$ . Consider that canonical ring homomorphism

$$\varphi : \text{End}_{\mathbf{R}}(E) \longrightarrow \text{End}_{\mathbf{S}}(\mathbf{S} \otimes E).$$

For  $\mathbf{f} \in \text{End}_{\mathbf{R}}(E)$ , define

$$\text{tr}(\mathbf{f}) = \text{tr}_{\mathbf{S}}(\varphi(\mathbf{f})),$$

where  $\text{tr}_{\mathbf{S}}$  is the usual trace of a matrix representation. It is independent of the chosen  $\mathbf{S}$ -basis of  $\mathbf{S} \otimes E$ .

To show that  $\text{tr}(\mathbf{f}) \in \mathbf{R}$  uses a standard argument. For each prime ideal  $\mathfrak{p}$  of  $\mathbf{R}$  of height 1,  $E_{\mathfrak{p}}$  is a free  $\mathbf{R}_{\mathfrak{p}}$ -module of rank  $r$ , and picking one of its basis gives also a  $\mathbf{S}$ -basis for  $\mathbf{S} \otimes E$  while showing that  $\text{tr}(\mathbf{f}) \in \mathbf{R}_{\mathfrak{p}}$ . Since  $\mathbf{R}$  is normal,  $\bigcap_{\mathfrak{p}} \mathbf{R}_{\mathfrak{p}} = \mathbf{R}$ , and thus  $\text{tr}(\mathbf{f}) \in \mathbf{R}$ .

Note that this defines an element of  $\text{Hom}_{\mathbf{R}}(\mathbf{C}, \mathbf{R})$ . Now looking at the isomorphism (5.1.1), yields that the trace of  $\mathbf{C}$  as an  $\mathbf{R}$ -module is contained in the trace ideal of  $E$ , which is a proper ideal of  $\mathbf{R}$ . But this is impossible since  $\text{tr}(\mathbf{I}) = r$ , which is a unit of  $\mathbf{R}$  by assumption. □

**Corollary 5.2.** *Let  $\mathbf{R}$  be a Noetherian local ring and  $E$  a finite  $\mathbf{R}$ -module.*

- (1) *If  $E$  is a local module and  $\mathbf{C} = \text{End}(E)$  has finite global dimension then  $E$  and  $\mathbf{C}$  are maximal Cohen-Macaulay  $\mathbf{R}$ -modules.*
- (2) *Moreover, if  $\mathbf{R}$  is integrally closed and  $E$  is a maximal Cohen-Macaulay module and the rank  $r$  of  $E$  is invertible in  $\mathbf{R}$  then  $\mathbf{R}$  is a regular local ring.*

*Proof.* Since  $\mathbf{C}$  is a local ring, by [4, Theorem 2.5] or [16, Theorem 3.1], is a Cohen-Macaulay  $\mathbf{R}$ -module. On the other hand,  $E$  is a  $\mathbf{C}$ -module and thus if  $\mathbf{x} = x_1, \dots, x_n$

is a maximal  $\mathbf{R}$ -sequence,  $n = \text{gl dim } \mathbf{C}$ ,  $E/(\mathbf{x})E$  is a  $\mathbf{C}$ -module with nonzero socle. It follows from Auslander-Buchsbaum formula (still valid for these algebras) that

$$\text{proj dim}_{\mathbf{C}} E/(\mathbf{x})E = \text{proj dim}_{\mathbf{C}} E + n = \text{gl dim } \mathbf{C} = n.$$

Thus by Theorem 5.1(ii)  $E$  is a free  $\mathbf{R}$ -module. This means that  $\mathbf{C}$  is a matrix ring over  $\mathbf{R}$  of finite global dimension and therefore  $\mathbf{R}$  is a regular local ring.  $\square$

**Remark 5.2.** Several extensions are suggested, the first on whether characteristic zero is required. Another is the possibility that the weaker condition  $\text{proj dim}_{\mathbf{C}}(E) < \infty$  will suffice. We have verified a special case in projective dimension one.

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